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The solution of linear Volterra Integral Equation of the first kind with Z-Transformation

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Abstract

When the advance problems of biology chemistry, physics and engineering field represent mathematically then linear voltera equations of first kind appear. In the present research, we used z-transform for solving linear Volterra integral equations of first kind. The convolution theorem for the z-transform has been proved. To demon state the effectiveness of z-transform for solving linear Volterra integral equation of first kind, some applications are given in application section.

Keywords: Linear Volterra integral equation of first kind, z-transform, Invers z-transform, Integral Equation.

حل معادلات فولتيرا التكاملية الخطية من النوع الأول باستخدام تحويل –Z

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الملخص

معادلات فولتيرا التكاملية من النوع الأول تظهر في العديد من مشاكل الهندسة، والفيزياء، والكيمياء، والأحياء. في هذا البحث يستخدم تحويل–Z للحصول على حل معادلات فولتيرا التكاملية الخطية من النوع الأول، ثم إثبات مبرهنة الالتفاف، وكذلك حل بعض المشاكل باستخدام تحويل–Z، وهذا يوضح فعالية وأهمية هذا التحويل لحل معادلات فولتيرا التكاملية من الدرجة الأولى.

الكلمات المفتاحية: معادلات فولتيرا التكاملية الخطية من النوع الأول، تحويل –Z، معكوس تحويل –Z، المعادلات التكاملية.

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Introduction

z-transform is used in many areas of applied mathematics as digital signal processing, control theory, economics and some other fields.

Recently there has been a great interest in studying integral equation methods and their applications. The advantage of the integral equation is witnessed by the increasing frequency of integral equation in the literature and in many filed, since more problems have their mathematical representation appear directly, and in a very natural way, in terms of integral equations. Other problems, whose direct representation is in terms of differential equations have their auxiliary conditions replaced by integral equations more elegantly than the differential equations. The name integral equation was introduced by Bois-Reymond in 1888. However the linear integral equation which is Volterra equation, was introduced by Volterra in 1884 integral transformations are encountered in many fields of engineering and science such as electrical networks, heat transfer mixing problems, signal processing, bending of beams newtons second law of motion, carbon dating problems, decay and exponential growth problems in later times, many the scientist are related in solving the problems of engineering and science by integral transform, the z-transform was first presented by Zain UI Abadin Zafar in 2016 [18]. In later times, many researches related in using z-transform [R.Salih, 2006, 51] is presented to solve the linear integral equation of convolution type. [E. Hussain, A. Jasim 2021, 32] is to study z-transform to solve non -linear difference equation. Due to this lmportant feature of the integral transforms many researches are attracted to this field and engaged in introducing various integral transform Laplace, Sumudu, Aboodh, Elzaki and Mahagoub transforms are compared and used it for solving boundary value problems by Dinkar Patia [17]. D. P. Patilet al [5,6] solved Volterra Integral equation of first kind by using different integral transform. Rathi sisters and D. P. Patil [15] solved system of differential equation by using Soham transform. Kushare transform is used for solving Volterra Integro – Differential equations of first kind by shinde, et al.

This paper is organized as follows: Linearity property of z-transform the first section. Second section is for z-transform of some Elementary function. Third section is Existence of z-transform. Fourth section is convolution Theorem for z-transforms. Fifth section is Inverse of z-transform. Sixth section is devoted to application of z-transform for solving Volterra integral equations.

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Definition 1 [6]

The linear Volterra integral equation of the first kind is given by

$f(t) = \int_0^x k(x,t)u(t)dt$; $u(x)$ is the unknown function and occurs only inside the integral sign. The function $f(x)$ and the kernel $k(x,t)$ are real – valued functions. The z-transform of the function $f(t)$ for $t \geq 0$ is defined as,

$$z(u,s) = z\{f(t)\} = s\int_0^\infty f(u\,t)e^{-st}dt \quad \ldots\ldots (1)$$

Or

$$z(u,s) = z\{f(t)\} = \frac{s}{u}\int_0^\infty f(t)e^{-\frac{s}{u}t}dt \quad \ldots\ldots. (2)$$

Where $z(u,s)$is transform operator. Assuming that the integral on the right side in (2) exists. The unique function $f(t)$ in (2) is called the inverse transform of $z(u,s)$ is indicated by

$$f(t) = z^{-1}\{z(u,s)\} \ldots\ldots. (3)$$

If $F(t)$ is piecewise continuous and of exponential order, the z-transform of the function $F(t)$ for $t \geq 0$ exist.

These conditions are only sufficient conditions for the existence of z-transform of the function $F(t)$.

1- Linearity property of z-transform.

If $z\{F(t)\} = A(u,s)$ and $z\{G(t)\} = B(u,s)$ then

$$z\{a\,F(t) + b\,G(t)\} = a\,z\{F(t)\} + b\,z\{G(t)\}$$

$$= a\,A(u,s) + b\,B(u,s)$$

Where a, b are arbitrary constants.

2- z-transform of some Elementary function.

NO	$f(t)$	*Z f(t)*
1	*1*	*1*

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2	t	$\frac{u}{s}$
3	e^{at}	$\frac{u}{s-au}$
4	$\sin at$	$\frac{aus}{s^2+a^2u^2}$
5	$\cos at$	$\frac{s^2}{s^2+a^2u^2}$
6	t^n	$n!\,\frac{u^n}{s^n}$
7	$e^{at}\sin bt$	$\frac{b\,\frac{s}{u}}{(\frac{s}{u}-a)^2+b^2}$
8	$e^{at}\cos bt$	$\frac{\frac{s^2}{u^2}-\frac{as}{u}}{(\frac{s}{u}-a)^2+b^2}$
9	$t\cos at$	$\frac{\frac{s}{u}(\frac{s^2}{u^2}-a^2)}{(\frac{s^2}{u^2}+a^2)^2}$
10	$t\sin at$	$\frac{2a\,\frac{s^2}{u^2}}{(\frac{s^2}{u^2}+a^2)^2}$

3- Existence of z-transform.

Theorem 1

If $f(t)$ is piecewise continuous in interval $0 \le t \le k$ and of exponential order γ for $t > k$, then its z- transform $z(u,s)$ exists for all $s > \gamma$, $u > \gamma$.

Proof.

We have for every positive number k.

$$\frac{s}{u}\int_0^\infty f(t)e^{\frac{-s}{u}t}dt = \frac{s}{u}\int_0^k f(t)e^{\frac{-s}{u}t}dt + \frac{s}{u}\int_k^\infty f(t)e^{\frac{-s}{u}t}\,dt$$

Since $f(t)$ is piecewise continuous in every finite interval $0 \le t \le k$, the first integral on the right side exists.

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Also, the second integral on the right side exists. So $f(t)$ is of exponential order γ for $t > k$.

To see this, we have only to observe that in such case:

$$\left[\frac{s}{u}\int_k^\infty f(t)e^{\frac{-s}{u}t}\,dt\right] \le \frac{s}{u}\left[f(t)e^{\frac{-s}{u}t}\right]dt$$

$$\le \frac{s}{u}\int_0^\infty e^{\frac{-s}{u}t}\big(f(t)\big)dt \le \frac{s}{u}\int_0^\infty e^{\frac{-s}{u}t} M\, e^{\gamma t}\,dt$$

$$\le \frac{sM}{u}\int_0^\infty e^{-\left(\frac{s}{u}-\gamma\right)t}\,dt$$

$$= \frac{sM}{u}\frac{e^{-\left(\frac{s}{u}-\gamma\right)t}}{\left(-\frac{s}{u}-\gamma\right)}\Bigg|_0^\infty = \frac{sM}{s-\gamma u}$$

Definition 2 (convolution of two functions).

Convolution of $F(t)$ and $G(t)$ functions is defined by

$$F(t) \otimes G(t) = F \otimes G$$

$$\int_0^t F(x)G(t-x)dx = \int_0^t F(t-x)G(x)dx\,.$$

$$\frac{u}{s}\frac{s^2}{s^2+u^2} z\{x(t)\} = 2\,\frac{s^2}{u^2}\frac{u^4}{(s^2+u^2)^2}$$

4- convolution Theorem for z-transforms.

Theorem 2.

If $z\{F(t)\} = A(u,s)$ and $Z\{G(t)\} = B(u,s)$ then:

$$z(f\otimes g) = \frac{u}{s}\, z(f)z(g)$$

$$z\{F(t) \otimes G(t)\} = \frac{u}{s}\, z\{F(t)\}\, z\{G(t)\} = \frac{u}{s}\, A(u,s)\, B(u,s)$$

Proof

$$z(f)z(g) = \frac{s}{u}\int_0^\infty f(\mathcal{T})e^{\frac{-s}{u}\mathcal{T}}d\mathcal{T}\int_0^\infty g(\theta)e^{\frac{-s}{u}\theta}d\theta$$

$$z(f)z(g) = \frac{s^2}{u^2} \int_0^{\infty} f(\mathcal{T})e^{\frac{-s}{u}\mathcal{T}} d\mathcal{T} \int_0^{\infty} g(\theta)e^{\frac{-s}{u}\theta} d\theta \quad \ldots\ldots\ (4)$$

$$t = \theta + \mathcal{T} \quad \text{and} \quad \theta = t - \mathcal{T}$$

$$z(g) = \int_{\mathcal{T}}^{\infty} g(t-\mathcal{T})e^{\frac{-s}{u}(t-\mathcal{T})} dt$$

$$= \int_{\mathcal{T}}^{\infty} g(t-\mathcal{T})e^{\frac{-s}{u}t}\, e^{\frac{s}{u}\mathcal{T}} dt$$

$$= e^{\frac{s}{u}z} \int_{\mathcal{T}}^{\infty} g(t-\mathcal{T})e^{\frac{-s}{u}t} dt$$

Thus

$$= \frac{s^2}{u^2} \int_0^{\infty} f(\mathcal{T})e^{\frac{-s}{u}\mathcal{T}} d\mathcal{T}\, e^{\frac{s}{u}\mathcal{T}} \int_{\mathcal{T}}^{\infty} g(t-\mathcal{T})e^{\frac{-s}{u}t} dt$$

$$= \frac{s^2}{u^2} \int_0^{\infty} f(\mathcal{T}) \int_{\mathcal{T}}^{\infty} e^{\frac{-s}{u}t}\, g(t-\mathcal{T})\, dt\, d\mathcal{T}$$

$$= \frac{s^2}{u^2} \int_0^{\infty} e^{\frac{-s}{u}t} \int_0^{t} f(t)\, g(t-\mathcal{T})\, d\mathcal{T}\, dt$$

$$= \frac{s^2}{u^2} \int_0^{\infty} e^{\frac{-s}{u}t}(f \otimes g)(t)dt = \frac{s}{u} z(f \otimes g)$$

$$z(f \otimes g) = \frac{u}{s} z(f)z(g)$$

5- Inverse of z-transform

If $z\{F(t)\} = z\{u, s\}$ then $F(t)$ is called the inverse z-transform of $z\{u, s\}$ and it is defined as $F(t) = z^{-1}\{z(u, s)\}$

where z^{-1} is the inverse z-transform operator.

6- Applications

In this chapter, some applications are given to show the effectiveness of z-transform for solving of linear Volterra integral equation of the first kind

Example 1.

Consider linear Volterra integral equation of the first kind

$$x = \int_0^{x} u(t)dt \quad \ldots\ldots\ (5)$$

Applying the z-transform to both sides of (5) we have

$$z\{x\} = z\left\{\int_0^x u(t)dt\right\} \quad \ldots\ldots\ldots\ldots (6)$$

Using convolution theorem of z-transform on (6), we have:

$$z\{\, x \,\} = \frac{u}{s}\, z\ \{\, 1 \,\} z\{\, u(x)\}$$

$$\frac{u}{s} = \frac{u}{s} \cdot 1 \cdot z\{\, u(x)\}$$

$$z\{\, u(x)\} = 1 \quad \ldots\ldots\ldots.. (7)$$

Operating inverse z-transform on both sides of (7), we have:

$$z^{-1}\left\{ z\{\, u(x)\}\right\} = z^{-1}\,(1)$$

$$u(x) = 1$$

This is the exact solution of equation (5)

Example 2.

Consider linear Volterra integral equation of the first kind:

$$x^2 = \frac{1}{2}\int_0^x (x-t)u(t)\, dt \quad \ldots\ldots. (8)$$

Applying the z-transform to both sides of (8) we have:

$$z\{\, x^2 \,\} = z\left\{\frac{1}{2}\int_0^x (x-t)u(t)dt\right\} \quad \ldots\ldots\ldots (9)$$

Using convolution theorem of z-transform on (9), we have:

$$2\,\frac{u^2}{s^2} = \frac{1}{2}\,\frac{u}{s}\, z\,\{x\} z\{u(x)\}$$

$$2\,\frac{u^2}{s^2} = \frac{1}{2}\,\frac{u}{s}\frac{u}{s}\, z\{u(x)\}$$

$$z\{\, u(x)\} = 4 \quad \ldots\ldots\ldots. (10)$$

Operating inverse z-transform on both sides of (10) we have:

$$z^{-1}\left\{ z\{\, u(x)\}\right\} = z^{-1}\{4\}$$

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$$u(x) = 4$$

This is the exact solution of equation (8)

Example 3.

Consider linear Volterra integral equation of the first kind.

$$y\,(t) = \; t^2 + \; \int_0^t y(u)\sin(t-u)du \quad \ldots\ldots\ldots.. (11)$$

$$z\,\{\,y(t)\} = z\,\{\,t^2 + \; \int_0^t y(u)\sin(t-u)du\,\}$$

From the linearity property of the inverse z-transform

$$z\,\{\,y(t)\} = z\,\{\,t^2\} + z\{\,\int_0^t y(u)\sin(t-u)du\,\} \quad \ldots\ldots\ldots.. (12)$$

$$z\,\{\,y(t)\} = z\,\{\,t^2\} + \frac{u}{s} z\,\{\,y(t)\}z\{\sin t\,\}$$

$$z\,\{\,y(t)\} = 2\,\frac{u^2}{s^2} + \frac{u}{s}\,z\,\{\,y(t)\}\frac{us}{s^2+u^2}$$

$$z\,\{\,y(t)\} - z\,\{\,y(t)\}\left(\frac{su}{s^3+su^2}\right) = 2\frac{u^2}{s^2}$$

$$z\,\{\,y(t)\}\left(1 - \frac{su^2}{s^3+su^2}\right) = 2\frac{u^2}{s^2}$$

$$z\,\{\,y(t)\}\left(\frac{s^3}{s^3+su^2}\right) = 2\frac{u^2}{s^2}$$

$$z\,\{\,y(t)\}\left(\frac{s^2}{s^2+u^2}\right) = 2\frac{u^2}{s^2}$$

$$z\,\{\,y(t)\} = 2\frac{u^2}{s^2}\cdot\frac{s^2+u^2}{s^2}$$

$$z\,\{\,y(t)\} = 2\frac{u^2}{s^2} + \frac{2u^4}{s^4} \quad \ldots\ldots\ldots.. (13)$$

Operating inverse z-transform on both sides of (13) we have:

$$z^{-1}\,\{\,z\{y(t)\}\} = \; z^{-1}\left\{\,2\,\frac{u^2}{s^2} + 2\,\frac{u^4}{s^4}\right\}$$

From the linearity property of the inverse z-transform

$$y(t) = z^{-1}\left\{2\frac{u^2}{s^2}\right\} + z^{-1}\left\{2\,\frac{u^4}{s^4}\right\}$$

$$y(t) = t^2 + \frac{t^4}{12}$$

This is the exact solution of equation (11)

Example 4.

Consider linear Volterra integral equation of the first kind.

$$\int_0^t \cos(t-s)x(s)ds = t\sin t \quad \text{........ (14)}$$

Applying the z-transform to both sides of (14), we have:

$$z\left\{\int_0^t \cos(t-s)x(s)ds\right\} = z\,\{t\sin t\} \quad \text{.......... (15)}$$

Using convolution theorem of z-transform on (15), we have:

$$\frac{u}{s}\,z\,\{\cos t\}\,z\,\{x(t)\} = \frac{2\frac{s^2}{u^2}}{(\frac{s^2}{u^2}+1)^2}$$

$$\frac{u}{s}\,\frac{s^2}{s^2+u^2}\;z\,\{x(t)\} = 2\,\frac{s^2}{u^2}\frac{u^4}{(s^2+u^2)^2}$$

$$z\,\{x(t)\} = 2\,\frac{us}{s^2+u^2} \quad \text{........ (16)}$$

Operating inverse z-transform on both sides of (16), we have:

$$z^{-1}\left\{z\,\{x(t)\}\right\} = z^{-1}\left\{2\,\frac{us}{s^2+u^2}\right\}$$

$$x(t) = \sin t$$

This is the exact solution of equation (14).

7- Discussion

As technology continues advancing, the need for efficient signal processing has become increasingly important. Transform methods provide an effective way to analyzes: gnarls and systems in different domains. Two of the most powerful and widely used transform methods are Laplace transform and z-transform.

However, while they may seem, important differences between these two methods make them better suited for different applications.

In this section, we well explore why we use z-transform instead Laplace transform to solve

1- The z-transform is better suited for analyzing the stability of discrete-time system than the Laplace transform. It because the z-transform provides away to analyze the poles and zeros of the transfer of a system.
2- The z-transform is better suited for analyzing finite length signals than the Laplace transform which means it can handle signals with a finite duration. In contrast, the Laplace transform is an infinite-length transform.
3- The z-transform is closely related to the sampling theory, the basis for digital signal processing therefore, when working with digital signals, the z-transform is often preffer over the Laplace transform.
4- The z-transform handles discrete-time signals whereas the Laplace transform best suits continuous method for analyzing these signals.

Conclusion

In the present paper, we have success fully defined the z-transform for solving linear Volterra integral equation of first kind. The given application showed that very less comulational work and a very little time needed for finding the exact solution of Linear Volterra integral equations of first kind. In future, z-transform can be applied for solving the system of linear Volterra integral equations.

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